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Positive Solutions to a Class of Elliptic Boundary Value Problems

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We prove the existence of positive solutions to the boundary value problems

$$\Delta u + \lambda a(x) f(u) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in R^N, a: $\Omega \to R$ may change sign, $f(0) > 0$, and $\lambda > 0$ is sufficiently small. Our approach is based on the Leray–Schauder fixed point theorem. © 1998 Academic Press

1. INTRODUCTION

Consider the problem

$$\begin{aligned} \Delta u + \lambda a(x) f(u) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in R^N with boundary $\partial\Omega$ and λ is a positive parameter.

Problem (1.1) with $a \geq 0$, $f \geq 0$ has been studied extensively (see, e.g., [5] and the references therein). We are motivated by a recent paper [1], in

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which the existence of a positive solution to (1.1) when a may change sign was established for $f(0) > 0$, λ small, and Ω a ball. In this note, we shall extend the result in [1] to the case of an arbitrary domain Ω, under less stringent assumptions on f. Our approach is based on the Leray–Schauder fixed point theorem. Note that if $f(0) = 0$ and f is sufficiently smooth, then by results in [3] and [4] there is bifurcation for positive solutions of (1.1) from the line of trivial solutions as long as $a(x) > 0$ for some $x \in \Omega$.

We make the following assumptions:

(A.1) $f\colon R^+ \to R$ is continuous and $f(0) > 0$.

(A.2) a is continuous on $\overline{\Omega}$, $a \not\equiv 0$, and there exists a number $k > 1$ such that

$$\int_\Omega K(x,y)a^+(y)\,dy \ge k\int_\Omega K(x,y)a^-(y)\,dy$$

for every $x \in \Omega$, where a^+ (resp. a^-) is the positive (resp. negative) part of a, and $K(x, y)$ is the Green's function of $-\Delta$ with Dirichlet boundary conditions.

Our main result is

THEOREM 1.1. *Let* (A.1)–(A.2) *hold. Then there exists a positive number* λ^* *such that* (1.1) *has a positive solution for* $\lambda < \lambda^*$.

As a consequence of Theorem 1.1, we have the following result for the radial Laplacian in a ball:

COROLLARY 1.2. *Let f satisfy* (A.1) *and let a be a nonzero integrable function on* $[0,1]$. *Suppose that there exists a number $k > 1$ such that*

$$\int_0^t s^{N-1}a^+(s)\,ds \ge k\int_0^t s^{N-1}a^-(s)\,ds$$

for every $t \in [0,1]$. *Then there exists a positive number* λ^* *such that the problem*

$$u'' + \frac{N-1}{t}u' + \lambda a(t)f(u) = 0, \qquad 0 < t < 1,$$

$$u'(0) = u(1) = 0$$

has a positive solution for $0 < \lambda < \lambda^*$.

Remark 1.1. Corollary 1.2 was established in [1] under the additional assumption that f is nondecreasing.

2. PROOFS

Throughout the paper, we assume that $f(u) = f(0)$ for $u \le 0$. We first need

LEMMA 2.1. *Let* $0 < \delta < 1$. *Then there exists a positive number* $\bar{\lambda}$ *such that, for* $0 < \lambda < \bar{\lambda}$, *the problem*

$$\begin{aligned} \Delta u &= -\lambda a^{+}(x) f(u) \qquad \textit{in } \Omega, \\ u &= 0 \qquad \textit{on } \partial\Omega \end{aligned} \tag{2.1}$$

has a positive solution $\tilde{u}_\lambda$ *with* $|\tilde{u}_\lambda|_0 \to 0$ *as* $\lambda \to 0$, *and*

$$\tilde{u}_\lambda(x) \ge \lambda \delta f(0) p(x), \qquad x \in \Omega,$$

where $p(x) = \int_\Omega K(x, y) a^{+}(y)\, dy$.

Proof. For each $u \in C(\bar{\Omega})$, let

$$Au(x) = \lambda \int_\Omega K(x, y) a^{+}(y) f(u(y))\, dy, \qquad x \in \bar{\Omega}.$$

Then $A: C(\bar{\Omega}) \to C(\bar{\Omega})$ is completely continuous and fixed points of A are solutions of (2.1). We shall apply the Leray–Schauder fixed point theorem to prove that A has a fixed point for λ small. Let $\varepsilon > 0$ be such that

$$f(x) \ge \delta f(0) \qquad \text{for } 0 \le s \le \varepsilon.$$

Suppose that $\lambda < \varepsilon / 2|p|_0 \tilde{f}(\varepsilon)$, where $\tilde{f}(t) = \max_{0 \le s \le t} f(s)$. Then there exists $A_\lambda \in (0, \varepsilon)$ such that

$$\frac{\tilde{f}(A_\lambda)}{A_\lambda} = \frac{1}{2\lambda |p|_0}.$$

Let $u \in C(\bar{\Omega})$ and $\theta \in (0,1)$ be such that $u = \theta A u$. Then we have

$$|u|_0 \le \lambda |p|_0 \tilde{f}(|u|_0)$$

or

$$\frac{\tilde{f}(|u|_0)}{|u|_0} \ge \frac{1}{\lambda |p|_0},$$

which implies that $|u|_0 \ne A_\lambda$. Note that $A_\lambda \to 0$ as $\lambda \to 0$. By the Leray–Schauder fixed point theorem, A has a fixed point $\tilde{u}_\lambda$ with $|\tilde{u}_\lambda|_0 \le A_\lambda < \varepsilon$. Consequently, $\tilde{u}_\lambda(x) \ge \lambda \delta f(0) p(x)$, $x \in \Omega$, and the proof is complete.

Proof of Theorem 1.1. Let $q(x) = \int_\Omega K(x,y)a^-(y)\,dy$. By (A.2), there exist positive numbers $\alpha, \gamma \in (0,1)$ such that

$$q(x)|f(s)| \le \gamma p(x) f(0) \tag{2.2}$$

for $s \in [0, \alpha]$, $x \in \Omega$. Fix $\delta \in (\gamma, 1)$ and let $\lambda^* > 0$ be such that

$$|\tilde{u}_\lambda|_0 + \lambda\delta f(0)|p|_0 \le \alpha \tag{2.3}$$

for $\lambda < \lambda^*$, where $\tilde{u}_\lambda$ is given by Lemma 2.1, and

$$|f(x) - f(y)| \le f(0)\left(\frac{\delta - \gamma}{2}\right) \tag{2.4}$$

for $x, y \in [-\alpha, \alpha]$ with $|x - y| \le \lambda^*\delta f(0)|p|_0$.

Let $\lambda < \lambda^*$. We look for a solution u_λ of (1.1) of the form $\tilde{u}_\lambda + v_\lambda$. Thus v_λ solves

$$\begin{aligned} \Delta v_\lambda &= -\lambda a^+(x)\big(f(\tilde{u}_\lambda + v_\lambda) - f(\tilde{u}_\lambda)\big) + \lambda a^-(x) f(\tilde{u}_\lambda + v_\lambda) \qquad \text{in } \Omega, \\ v_\lambda &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

For each $w \in C(\overline{\Omega})$, let $v = Aw$ be the solution of

$$\begin{aligned} \Delta v &= -\lambda a^+(x)\big(f(\tilde{u}_\lambda + w) - f(\tilde{u}_\lambda)\big) + \lambda a^-(x) f(\tilde{u}_\lambda + w) \qquad \text{in } \Omega \\ v &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Then $A\colon C(\overline{\Omega}) \to C(\overline{\Omega})$ is completely continuous. Let $v \in C(\overline{\Omega})$ and $\theta \in (0,1)$ be such that $v = \theta A v$. Then we have

$$\Delta v = -\lambda\theta a^+(x)\big(f(\tilde{u}_\lambda + w) - f(\tilde{u}_\lambda)\big) + \lambda\theta a^-(x) f(\tilde{u}_\lambda + w).$$

We claim that $|v|_0 \ne \lambda\delta f(0)|p|_0$. Suppose to the contrary that $|v|_0 = \lambda\delta f(0)|p|_0$. Then, by (2.3) and (2.4), we obtain

$$|\tilde{u}_\lambda + v|_0 \le |\tilde{u}_\lambda|_0 + |v|_0 \le \alpha$$

and

$$|f(\tilde{u}_\lambda + v) - f(\tilde{u}_\lambda)|_0 \le f(0)\frac{\delta - \gamma}{2},$$

which, together with (2.2), implies that

$$\begin{aligned} |v(x)| &\le \lambda\frac{\delta - \gamma}{2} f(0)p(x) + \lambda\gamma f(0)p(x) \\ &= \lambda\frac{\delta + \gamma}{2} f(0)p(x), \qquad x \in \Omega. \end{aligned} \tag{2.5}$$

In particular

$$|v|_0 \le \lambda \frac{\delta + \gamma}{2} f(0)|p|_0 < \lambda \delta f(0)|p|_0,$$

a contradiction, and the claim is proved. By the Leray–Schauder fixed point theorem, A has a fixed point v_λ with $|v_\lambda|_0 \le \lambda\delta f(0)|p|_0$. Hence v_λ satisfies (2.5) and, using Lemma 2.1, we obtain

$$\begin{aligned} u_\lambda(x) &\ge \tilde{u}_\lambda(x) - v_\lambda(x) \\ &\ge \lambda\delta f(0) p(x) - \lambda \frac{\delta + \gamma}{2} f(0) p(x) = \lambda \frac{\delta - \gamma}{2} f(0) p(x), \end{aligned}$$

i.e., u_λ is a positive solution of (1.1). This completes the proof of Theorem 1.1. ∎

Proof of Corollary 1.2. By integrating, it follows that the solution of

$$u'' + \frac{N-1}{t} u' = -a^{\pm}(t), \qquad 0 < t < 1,$$
$$u'(0) = u(1) = 0$$

is given by

$$u^{\pm}(t) = \int_t^1 \frac{1}{S^{N-1}} \left(\int_0^s \tau^{N-1} a^{\pm}(\tau)\, d\tau \right) ds.$$

This implies that $u^+ \ge k u^-$, and since integrability of a is sufficient for the radial Laplacian, the result follows from Theorem 1.1. ∎

REFERENCES

[1] N. P. Cac, A. M. Fink, and J. A. Gatica, Nonnegative solutions of the radial Laplacian with nonlinearity that changes sign, *Proc. Amer. Math. Soc.* **123** (1995), 1393–1398.

[2] R. Courant and D. Hilbert, "Methods of Mathematical Physics," Vol. 2, Interscience, New York, 1953.

[3] P. Hess, On bifurcation and stability of positive solutions of nonlinear elliptic eigenvalue problems, *in* "Dynamical Systems II," (Bednarek and Cesari, eds.), pp. 103–119, Academic Press, San Diego, 1982.

[4] P. Hess and T. Kato, On some linear and nonlinear eigenvalue problems with an indefinite weight function, *Comm. Partial Differential Equations* **5** (1980), 999–1030.

[5] P. L. Lions, On the existence of positive solutions of semilinear elliptic equations, *SIAM Rev.* **24** (1982), 441–467.